

Zero-range processes with saturated condensation: the steady state and dynamics

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Abstract. We study a class of zero-range processes in which the real-space condensation phenomenon does not occur and is replaced by a saturated condensation: that is, an extensive number of finite-size “condensates” in the steady state. We determine the conditions under which this occurs, and investigate the dynamics of relaxation to the steady state. We identify two stages: a rapid initial growth of condensates followed by a slow process of activated evaporation and condensation. We analyze these nonequilibrium dynamics with a combination of mean-field approximations, first-passage time calculations and a fluctuation-dissipation type approach.

1. Introduction

Real-space condensation, whereby a finite fraction of a system’s mass accumulates within a microscopic region, is a spectacular phenomenon that is observed in a wide range of dynamical systems. For example, it is manifested experimentally in shaken granular gases in which particles can diffuse between compartments [1, 2]: as the driving strength is reduced, the sand grains cluster into a single compartment. One can also find examples in models of macroeconomics, whereby a large fraction of the available wealth is accumulated by a single individual [3], and of traffic flow, in which buses serving a single route cluster together [4].

One of the requirements for a thermodynamic condensation transition to occur as the total density of particles is increased is the absence of any restriction on the total number of particles that may occupy a single site. In this work, we are interested in the case where such unbounded growth of particle number is inhibited. This can happen quite naturally within specific applications: for example, the compartments in the granular gas experiments of [1, 2] are of finite size, and once they contain more than a certain number of particles, any extra particles may diffuse freely out of them. As we will show below, a vestige of the condensation phenomenon may still be observed in the form of a separation of the system into high- and low-density sites: we call this *saturated condensation*. The key questions then are: (i) Under what conditions is saturated condensation observed? (ii) How is the state of saturated condensation approached dynamically from some given initial condition?

In this work, we will provide answers to both these questions with reference to a specific well-studied mathematical model, the zero-range process (ZRP, see [5, 6] for reviews). This is a stochastic dynamical model in which particles occupy sites of a lattice (or network) and hop to neighbouring sites at a rate that depends only on the number of particles on the departure site: it is for this reason that it is described as ‘zero-range’. The model was introduced and solved for its steady-state behaviour by Spitzer [7], and it was realized a little over ten years ago that real-space condensation [8] is possible at sufficiently large particle densities under certain conditions on the hop rates [4], even within a spatially homogeneous system (see also [5, 6]). The dynamics of condensation onto a single site has also been of interest. Here, the focus has been on the late-time coarsening of the excess mass into a decreasing number of increasingly massive clusters, which in a finite system ends with a process of mass exchange between the last remaining clusters [9, 10]. In the steady state, for a finite system, the condensate occasionally melts and reforms on a new site: the timescale of this process has been the subject of some discussion [10, 11, 12].

Variants of the ZRP where mass accumulates on multiple sites—and in particular an extensive number of sites—are comparatively little studied. Schwarzkopf et al [13] examined the statics and dynamics of a ZRP with transition rates chosen in such a way that a single condensate is destabilized in favour of either a finite number of extensive condensates, or a subextensive number of subextensive ‘mesocondensates’. In both

cases, one still has a finite fraction of the mass occupying a vanishingly small fraction of the sites in the thermodynamic limit, and thus a true condensation transition is observed in this model.

By contrast, we consider here the case where the hop rates in the ZRP are chosen such that the mass of condensates reaches a finite size that does not increase with system size. This prevents a true condensation transition, but nevertheless admits the possibility of saturated condensation discussed above. Since the true condensation limit can be approached by taking the upper limit on the size of a condensate to infinity, it seems clear that saturated condensation can have interesting consequences at a phenomenological level even if no formal singularity remains. This is similar to the equilibrium phenomenon of “micellization” in which attractive particles can form clusters whose size is limited by their packing geometry [14]. An analysis of this problem as an instance of saturating condensation in equilibrium was offered by Goldstein [15].

After recalling the definition of the ZRP in section 2 and briefly reviewing the conditions for a condensation transition in the homogeneous system (section 2.1), we present in section 2.2 the conditions on the ZRP hop rates for saturated condensation to occur. We then turn to the dynamics of the process. One can first ask about the dynamics within the steady state. This involves evaporation and formation of condensates, and in section 2.3 we calculate the rates at which both processes take place. The remainder of our work, presented in section 3, concerns an investigation of the relaxation to the steady state from a prescribed initial condition. Our main finding is that this is a nontrivial, two-stage process. First, mass rapidly accumulates on sites that are selected by the local dynamics and in a way that depends on the initial condition. The number of such sites typically differs from its global equilibrium value. This gives rise to a much slower second stage in which condensates are nucleated and evaporate as activated processes. We obtain a detailed portrait of both stages of the relaxational dynamics through a combination of mean-field theory, calculation of first-passage properties and an approach reminiscent of a fluctuation-dissipation analysis. In certain cases, these approximations agree remarkably well with stochastic simulations. These analytical results thus constitute a more complete account of a nonequilibrium condensation dynamics that has been achieved so far.

2. Presentation of the model: Steady state and condensation

The model we consider is defined on a one-dimensional lattice of L sites with periodic boundary conditions. Each site i can be occupied by an arbitrary number of particles n_i . Since the system is not connected to any reservoirs, the total number of particles in the system $N = \sum_i n_i$ is constant. A particle can move from site i to a neighbouring site $i \pm 1$ with rates $u_i^\pm(n_i)$, respectively (see figure 1). We call $v_i^\pm(n)$ the hopping rate *per particle* so that $u_i^\pm(n_i) = n_i v_i^\pm(n_i)$. Qualitatively, our main results hold for both symmetric and asymmetric hoping rates but to avoid redundancies we shall only present the symmetric case. The definition of the zero-range process (ZRP) is that the hopping

rates depend only on the number of particles at the *departure* site and not, for instance, on the occupancy of the target site. In general the rates could also vary from site to site but in this work we consider only spatially homogeneous systems.

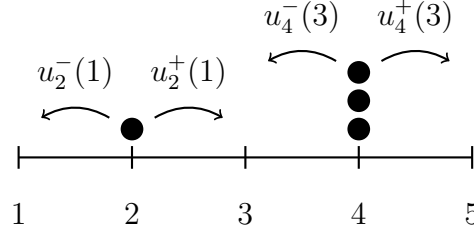


Figure 1. Presentation of the model. The arrows indicate the allowed transitions and the rates at which they occur.

2.1. Factorization and condensation

Although the ZRP has been extensively reviewed in the literature [5, 6] we shall briefly summarize some known results that will be important in our understanding of saturated condensation, deferring to these articles for further details. First, because the interactions between particles are limited to a single site, the steady-state distribution of occupancies factorizes. That is, in the canonical ensemble one has [6]

$$P(\{n_i\}) = \frac{1}{Z_{N,L}} \prod_{i=1}^L g(n_i) \delta\left(\sum_{j=1}^L n_j - N\right) \quad (1)$$

where the partition function $Z_{N,L}$ is given by

$$Z_{N,L} = \sum_{\{n_i\}} \prod_{i=1}^L g(n_i) \delta\left(\sum_{i=1}^L n_i - N\right) \quad (2)$$

and the factors $g(n_i)$ are determined by the hopping rates

$$g(n) = \prod_{j=1}^n \frac{1}{u(j)} \quad \text{for } n > 0 \quad \text{and} \quad g(0) = 1. \quad (3)$$

The delta functions in (1) and (2) simply enforce the constraint that the total number of particles on the lattice is fixed. The marginal probability that a given site has n particles is given by

$$p_i(n) = g(n) \frac{Z_{L-1, N-n}}{Z_{L,N}}. \quad (4)$$

As previously mentioned, the ZRP admits an interesting condensation transition. Although it can be worked out directly from the canonical ensemble [16], the condition for condensation is most easily seen in the grand canonical ensemble. Introducing a

chemical potential μ and the single-site partition function $\mathcal{Z}_1(\mu)$, the partition function for the L -site system reads

$$\mathcal{Z}_L(\mu) = \sum_N \exp(\mu N) Z_{L,N} = \left[\sum_n e^{\mu n - \sum_{j=1}^n \ln[u(j)]} \right]^L \equiv [\mathcal{Z}_1(\mu)]^L \quad (5)$$

Note that the only correlations between different sites in the canonical ensemble come from the constraint on the total number of particles in (1). Since this constraint has been removed in favour of a chemical potential, the sites are now completely uncorrelated in the steady state and the L -site partition function reduces to a single-site problem. To get more insight into \mathcal{Z}_1 , one can rewrite the hopping rate per particle as $v(j) = v_0 e^{h(j)}$, where v_0 is the hopping rate of a single particle and $e^{h(j)}$ encodes the interaction between the particles. For instance, $h(j) = 0$ for all j corresponds to non-interacting particles. The partition function then reads

$$\mathcal{Z}_1(\mu) = \sum_n \exp[-F(n, \mu)] \quad (6)$$

where $F(n, \mu) = f(n) - \mu n$ and we have introduced

$$f(n) = \ln(v_0) n + \ln(n!) + \sum_{j=1}^n h(j). \quad (7)$$

The marginal probability that a single site has n particles is then given by

$$p(n|\mu) = e^{\mu n} g(n) \frac{\mathcal{Z}_{L-1}(\mu)}{\mathcal{Z}_L(\mu)} \quad (8)$$

$$= \frac{1}{\mathcal{Z}_1(\mu)} \exp[\mu n - f(n)] \quad (9)$$

and the average number of particles per site by [6]

$$\langle n \rangle \equiv \rho = \frac{\mathcal{Z}'_1(\mu)}{\mathcal{Z}_1(\mu)}. \quad (10)$$

We thus see from (9) that $f(n)$ plays the role of a single-site free energy, and hence that $F(n, \mu)$ is a single-site grand canonical potential.

The general idea of ensemble equivalence is to ask what chemical potential μ should be imposed to get a given value of $\langle n \rangle$. To detect a possible condensation transition, one thus looks for the maximum density ρ_c for which equation (10) has a solution. If this maximum is infinite, then (10) is always solvable and there is no transition. If, however, ρ_c is finite then equation (10) cannot be solved for a density greater than ρ_c and the excess mass condenses on a single site, which can thus carry a finite fraction of the total mass of the system. The breaking of ensemble equivalence between canonical and grand canonical ensemble is a signature of the condensation transition.

The condition for (10) to have a solution has been worked out and yield a criterion on the form of $u(n)$ to observe condensation. It can be summarized as follows [6]:

- if $u(n)$ decays to a non-zero constant more slowly than $u(n) \simeq \beta(1 + 2/n)$, one observes above a non-zero critical density the appearance of a single condensate in a background fluid which remains at the critical density.

- if $u(n) \rightarrow 0$ as $n \rightarrow \infty$, condensation occurs at all densities and the fraction of particles in the fluid phase tends to zero.
- otherwise, and in particular if $u(n)$ increases as $n \rightarrow \infty$, condensation does not occur.

The first two cases, in which there exists a true thermodynamic phase transition, have previously received much attention in the literature (as reviewed in [6]). What we shall show in the following is that the third case also may also exhibit interesting condensation-like features, despite the absence of a true condensation transition, when the stationary state supports a coexistence of sites at two characteristic densities. We shall refer to this case as *saturated condensation*.

2.2. Criteria for saturated condensation

As mentioned in the Introduction, it is natural to expect condensation in real space to saturate at some large but finite value for the mass of the condensate (e.g., in shaken granular gases when the finite size of the compartments prevents true condensation). We will therefore consider systems where the hop rate per particle $v(n)$ asymptotically decreases to a finite but non-zero value. Even if the total hop rate per site, $u(n)$, initially decreases with n , it eventually starts increasing again and there is no phase transition, as discussed above. However, we will show that the initial decrease in $v(n)$ may suffice to destabilize a homogeneous state, whereas the asymptotic growth of $u(n)$ prevents the formation of condensates with a mass that diverges with the system size. One thus ends up with a steady-state containing an extensive number of finite-sized condensates. Since there is no thermodynamic transition, canonical and grand canonical ensembles remain equivalent. Therefore, we can perform our analysis solely within the latter ensemble, and dispense with the cumbersome constraint on the total number of particles. On the other hand, simulations are most straightforwardly conducted in the canonical ensemble. Since we shall use both ensembles in the following, we will refer equivalently to free energy or grand potential with the understood assumption that chemical potential μ and number of particles N are adjusted so that $\sum_n n p(n|\mu) = N$.

For the system to start forming condensates (i.e., high-density sites in a sense to be defined more formally below) one needs the single site free energy $f(n)$ to be non-convex, that is $f''(n) < 0$ for some range of n . Flat profiles with such occupancies would then be unstable under the dynamics and undergo spinodal decomposition. Treating n as a continuous variable, we have $f(n) \simeq \int_1^n \ln u(n') dn'$, and hence the second derivative with respect to n is given by

$$f''(n) \simeq \frac{u'(n)}{u(n)} \quad (11)$$

which implies spinodal decomposition for occupancies such that $u'(n) < 0$. This is exactly equivalent to the condition [17] that the hop rate per particle, $v(n) = u(n)/n$,

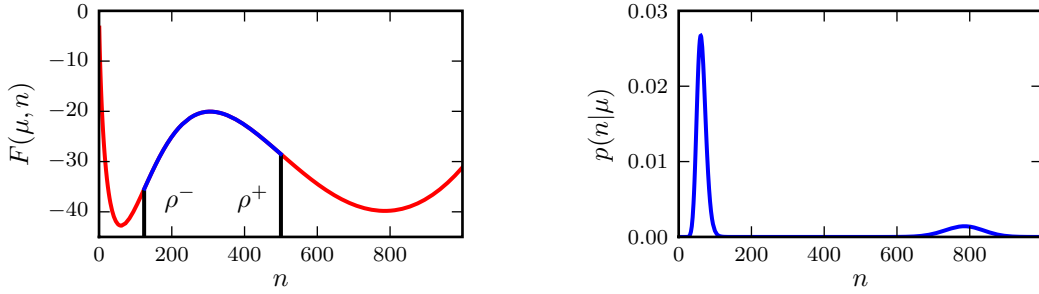


Figure 2. **Left:** The grand potential density per site, $F(\mu, n)$, for the choice $u(n) = v_0 n \exp(-\lambda\phi \arctan(n/\phi))$ discussed in the text, with $v_0 = 2.5$, $\lambda = 0.01$, $\phi = 250$ and $\langle n \rangle = 100$. The region unstable to spinodal decomposition is in blue: it corresponds to the concave part of the grand potential. **Right:** The resulting, normalized, probability distribution.

should satisfy the equation

$$v'(n) < \frac{v(n)}{n}. \quad (12)$$

Such an instability could in principle lead to complete condensation: the criterion (12) is indeed satisfied when the condensation transition occurs. This is because the phase separation can lead to phase coexistence between a low-density phase and a high-density phase whose density can diverge with system size, i.e., a macroscopic condensate. This is for instance what happens in [13], where the jump rates $u(n)$ depends on the system size L in such a way that the mass of each of the multiple condensates diverges in the thermodynamic limit. For the high density sites to have *finite* occupancies, we will further require that $f'(n) \rightarrow \infty$ when $n \rightarrow \infty$. Combined with the fact $f(n)$ is not everywhere convex this implies that the free energy per site has a double tangent between two finite densities [18]. Under these conditions the grand canonical potential per site, $F(n, \mu)$, forms a double well whose minima occur for finite values of n and give the typical occupancy of the high- and low-density sites. ‘Saturated condensation’ now is to be understood as referring to this scenario. In terms of the microscopic jump rates, the requirement that $f'(n) \rightarrow \infty$ is equivalent to $u(n) \rightarrow \infty$ as $n \rightarrow \infty$, so $u(n)$ must be an unbounded increasing function of n or, correspondingly, $v(n)$ must either increase as $n \rightarrow \infty$, or decrease more slowly than $1/n$.

Let us illustrate with a concrete example of the function $u(n)$ that leads to saturated condensation, and which we will use repeatedly throughout the remainder of this work. It reads

$$u(n) = v_0 n \exp[-\lambda\phi \arctan(n/\phi)]. \quad (13)$$

With this definition, the hop rate per particle $v(n)$ initially decreases exponentially with n before it saturates at a constant value $v_0 \exp(-\lambda\phi\pi/2)$. A flat profile is unstable if $u'(n) < 0$, that is if $\lambda\phi > 2$ and $n \in [\rho^-, \rho^+]$ where

$$\rho^\pm = \frac{\lambda\phi^2 \pm \phi\sqrt{\lambda^2\phi^2 - 4}}{2}. \quad (14)$$

As $n \rightarrow \infty$, $u(n) \sim v_0 n \exp(-\lambda \phi \pi/2)$ and is thus increasing linearly with n ; the condensates have finite size. This choice of rates gives rise to the double-well free energy and bimodal probability distribution shown in fig. 2.

Note that the range $[\rho^-, \rho^+]$ corresponds to the concave part of the free energy, as expected from standard thermodynamics [19]. As long as the average density lies between the two minima of $F(\mu, n)$, the steady state will be dominated by configurations with condensates. Flat profiles will however be metastable outside $[\rho^-, \rho^+]$, thus requiring activated events to lead to condensation. Simulations of the systems for rates obeying (13) show the predicted behaviour. On the left panel of figure 3 one sees the results of simulations started with N particles distributed randomly over the L sites. The average density $\rho = N/L$ is chosen either within or outside the condensation regime. The criteria for the condensate to have finite mass can be checked by considering the family of rates defined by

$$v_\alpha(n) = v_0 n^{-\alpha} \exp[-\lambda \phi \arctan(n/\phi)] \quad (15)$$

One indeed sees that the minimum in the grand potential corresponding to the high-density phase is at a finite value of n for $\alpha < 1$ and diverges when $\alpha \rightarrow 1$ (see right panel of figure 3).

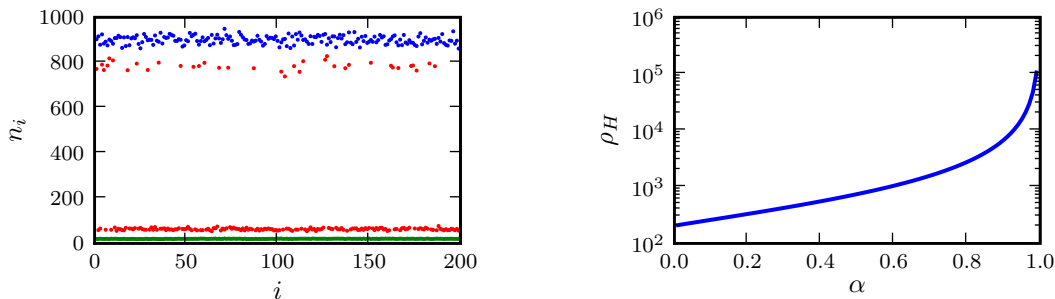


Figure 3. **Left:** Steady-states of stochastic simulations for $\phi = 250$, $\lambda = 0.01$, $v_0 = 2.5$. For these parameters, the minima in the grand potential correspond to $\rho_L \approx 60$ and $\rho_H \approx 800$. Occupancies are averaged over a time window $t \in [5000; 15000]$. Green and blue symbols correspond to initial average densities ($\rho = 20$ and $\rho = 900$ respectively) that are either below ρ_L or above ρ_H and are as expected stable. For an initial density $\rho = 300$ between ρ_L and ρ_H (red symbols), steady-state configurations typically exhibits a low density background at $\rho = \rho_L$ and high density condensates at $\rho = \rho_H$. **Right:** Semi-log plot of the typical mass of the high density phase for different value of α , using the rates (15). One sees that when $\alpha \rightarrow 1$, the mass of the condensate diverges as expected.

2.3. Condensation and evaporation dynamics in the steady state

When the average density lies between the two minima of the grand potential, there is a coexistence of high- and low-density sites in steady-state. Since the grand potential barrier between them is finite (see figure 2), the instantaneous number of condensates will fluctuate as low-density sites condense and high-density sites evaporate. To discuss

these processes in more detail, it is helpful to formally define a condensate (or high-density site) as a site with a density greater than the one at the peak in the grand potential between the two minima. Likewise, when the density on a site lies below the value corresponding to this peak, we refer it as a low-density site. With this definition, the average densities of the high- and low-density sites in the steady state are

$$\rho_L = \frac{\sum_{n=0}^{n_{\text{peak}}} n p(n|\mu)}{\sum_{n=0}^{n_{\text{peak}}} p(n|\mu)} \quad \text{and} \quad \rho_H = \frac{\sum_{n=n_{\text{peak}}}^{\infty} n p(n|\mu)}{\sum_{n=n_{\text{peak}}}^{\infty} p(n|\mu)} \quad (16)$$

while the average number of condensates in steady state is given by

$$n_c^{\text{eq}} = L \sum_{n=n_{\text{peak}}}^{\infty} p(n|\mu) = L \frac{\rho - \rho_L}{\rho_H - \rho_L} . \quad (17)$$

We may then also define the rate of evaporation of the condensates, $\mathcal{R}_{\text{evap}}$, and condensation of the low-density sites, $\mathcal{R}_{\text{cond}}$. At equilibrium these balance in such a way that the number and size of the condensates remain constant on average: $\mathcal{R}_{\text{evap}} n_c^{\text{eq}} = (L - n_c^{\text{eq}}) \mathcal{R}_{\text{cond}}$.

Although formation and evaporation of a condensate requires many hops and are thus complicated processes, the corresponding rates can be computed following a first-passage time approach [20]. Indeed if we know the first passage time $T_{n,n_{\text{peak}}}$ from a high density site $n \simeq \rho_H$ to $n = n_{\text{peak}}$, we can approximate the evaporation rate for a site with n particles by

$$\mathcal{R}_{\text{evap}}(n) = \frac{1}{2 T_{n,n_{\text{peak}}}} , \quad (18)$$

where the factor of a half arises as a site at the peak can fall in either direction with equal probability. To calculate the evaporation and condensation rates in the steady state, we must therefore calculate the relevant first-passage times for diffusion in a double well. To achieve this we follow previous approaches applied to ZRPs undergoing a thermodynamic condensation transition [12, 21].

We illustrate this procedure by computing the first-passage time to an evaporation event. We thus consider a high-density site with $n > n_{\text{peak}}$ particles. The rates at which the occupancy decreases or increases are given by

$$W(n \rightarrow n-1) = u(n) = 2 v_0 n e^{-\lambda \phi \arctan(\frac{n}{\phi})} \quad (19)$$

$$W(n \rightarrow n+1) \equiv u_L . \quad (20)$$

In principle, the rate at which particles are added on top of a condensate depends on the neighbouring densities. We will however assume their fluctuations to be small and consider u_L to be constant. The first-passage time from n particles to n_{peak} is denoted $T_{n,n_{\text{peak}}}$ and, in continuous time, is the solution to the equation

$$\begin{aligned} T_{n,n_{\text{peak}}} = dt + [1 - (u_L + u(n)) dt] T_{n,n_{\text{peak}}} \\ + u(n) dt T_{n-1,n_{\text{peak}}} + u_L dt T_{n+1,n_{\text{peak}}} \end{aligned} \quad (21)$$

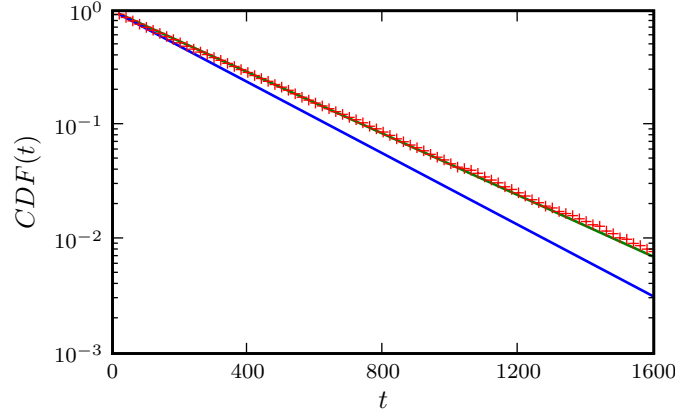


Figure 4. Semi-log plot of the cumulative distribution function of evaporation or condensation events in steady state for $\phi = 50$, $\lambda = 2.8/50$, $v_0 = 2.5$, $\langle n \rangle = 100$. The blue line corresponds to the cumulative distribution function of a Poisson distribution, $F_{CD}(t) = \exp(-\gamma t)$ with $\gamma \approx 0.0036$ as predicted by equation 24. The red dots stem from 10000 simulations and can be fitted with a rate $\gamma \approx 0.0031$ (green line).

Lengthy but standard algebra (see Appendix A) leads to

$$T_{n,n_{\text{peak}}} = \sum_{l=n_{\text{peak}}+1}^n \frac{1}{u(l)p(l|\mu)} \sum_{m=l}^{\infty} p(m|\mu). \quad (22)$$

Similarly one finds for the first passage time for condensation

$$T'_{n,n_{\text{peak}}} = \sum_{i=n+1}^{n_{\text{peak}}} \frac{1}{u_H p(i|\mu)} \sum_{j=0}^{i-1} p(j|\mu). \quad (23)$$

Note that to determine more accurately the rates of evaporation and condensation we should in principle average over all starting positions above and below the barrier, respectively:

$$T_{\text{evap}} = \frac{\sum_{n=n_{\text{peak}}}^{\infty} p(n|\mu) T_{n,n_{\text{peak}}}}{\sum_{n=n_{\text{peak}}}^{\infty} p(n|\mu)} \quad \text{and} \quad T_{\text{cond}} = \frac{\sum_{n=0}^{n_{\text{peak}}} p(n|\mu) T'_{n,n_{\text{peak}}}}{\sum_{n=0}^{n_{\text{peak}}} p(n|\mu)}. \quad (24)$$

We can then define the escape rate from a configuration as the rate for either an evaporation or condensation to occur:

$$\mathcal{R}_{\text{total}} = \frac{n_c}{2T_{\text{evap}}} + \frac{L - n_c}{2T_{\text{cond}}}. \quad (25)$$

In this picture, the distribution of times between events, either evaporations or condensations, for a system of length L will be Poissonian with rate $\mathcal{R}_{\text{total}}$. For the choice of parameters $\phi = 50$, $\lambda = 2.8$, $v_0 = 2.5$, $L = 5000$ and $\langle n \rangle = 100$, equation (25) can be evaluated numerically. First we compute μ so that $\langle n \rangle = 100$ by solving (10). We can then use expression (9) for $p(n|\mu)$ to compute T_{evap} and T_{cond} from (22), (23) and (24). We last obtain from (25) that the total rate is $\mathcal{R}_{\text{total}} \approx 0.0036$. To compare this theoretical prediction with numerics we compute the cumulative distribution function of evaporation and condensation events from 10000 runs: $F_{CD}(t)$ is the probability

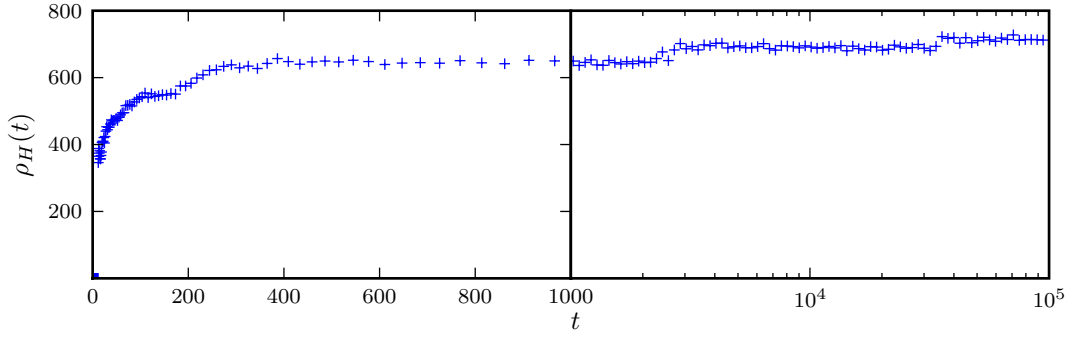


Figure 5. Average density in the high density sites as a function of time for parameters $\phi = 250$, $\lambda = 0.01$, $v_0 = 2.5$. The early dynamics ($t \sim 0 - 1000$) see the rapid formation of n_c condensates that are rapidly growing. The late stage dynamics $t \sim 10^3 - 10^5$ correspond to formation and evaporation of condensates that leads n_c to n_c^{eq} and the average mass of the condensates to its equilibrium value. Steps in the average density correspond to the evaporation of a condensate that is redistributed on the surviving ones. (Note the switch from linear to logarithmic scale on the time axis at $t = 1000$.)

that the first evaporation or condensation occurs after time t . The simulation data are shown in fig. 4; as predicted by the theory the distribution is Poissonian and a fit to the simulation data gives $\mathcal{R}_{\text{total}} \approx 0.0031$, which is within 10% of the predicted value.

3. Two-stage dynamics of condensate formation

We now examine the relaxation of the system to its steady-state, which is a nontrivial process. Starting from a homogeneous configuration within the unstable region ($n \in [\rho^-, \rho^+]$, see figure 2), the dynamics divides naturally into two regimes presented on figure 5. The early-time dynamics see the instability of the flat profile give birth to some number n_c of condensates that then rapidly grow. At the end of this growth stage, n_c is in general not equal to the equilibrium number n_c^{eq} and the system has not yet reached stationarity. A second stage then follows, taking place on much longer timescales, during which activated events responsible for condensation and evaporation of condensates lead the system towards its ultimate steady state. This difference in relevant timescales between the two stages can be seen from figure 5. Our aim in this section is to understand these two distinct relaxational regimes.

Simulations of the stochastic system, started from random deposition of N particles over the L sites of the lattice with $N/L \in [\rho^-, \rho^+]$, show that the positions of the condensates are, initially, anti-correlated (see figure 6). Transient anti-correlations of this type are a general feature of systems obeying a conservation law, see, for example, reference [22]. This can be understood as a consequence of the condensates being created through depletion of the neighbouring sites, thereby preventing the formation of other condensates in their immediate surroundings: if there is a condensate at site i , there is a decrease in the probability to find another condensate in its vicinity. These correlations survive until the late-stage dynamics when new condensates are formed and

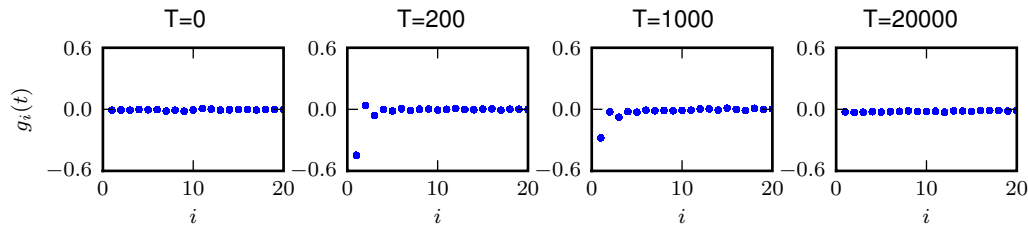


Figure 6. Snapshots of the correlation function $g_i(t) = \langle n_{j+i}(t)n_j(t) \rangle_c / \langle n_j^2(t) \rangle_c$, where $\langle x^2 \rangle_c \equiv \langle x^2 \rangle - \langle x \rangle^2$ and the averages are taken both over the lattice site j and many simulations. Starting from an initially flat profile, an anti-correlation between sites forms as the condensates condense which then gradually disappears at late times when subsequent evaporations and condensations randomize the positions of the condensates. The parameters of the simulation are $v_0 = 2.5$, $\Phi = 50$, $\lambda = .05$ and $\langle n \rangle = 80$.

old condensates evaporate, thus smoothing out the correlations \ddagger . Note that starting with the correct number of *regularly-spaced* condensates at the correct steady-state density would lead to smoothing of the correlations on the same timescales: correlations are mainly due to the immobility of the condensates, which only get randomized (by evaporation/condensation) in the late stage of the dynamics. Also, if one starts with a global density within $[\rho_L, \rho_H]$ but well outside $[\rho^-, \rho^+]$, there is no initial instability since the flat profile is metastable and the first stage is thus absent: the creation of condensates is then only due to activated events. We now turn to a more detailed analysis of both stages.

3.1. Initial instability and growth stage

The early stage dynamics corresponds to the rapid growth of an instability around the flat profile which leads to the formation of some number of condensates n_c . These condensates then rapidly grow and saturate at a density that generically differs from that at the minimum of the high density well in the free energy. Insight into this part of the dynamics can be gained by comparing the stochastic dynamics of the system with its deterministic mean-field limit. The latter is obtained by replacing $\langle u(n_i) \rangle$ by $u(\langle n_i \rangle) = u(\rho_i)$; it reads

$$\dot{\rho}_i = u(\rho_{i-1}) + u(\rho_{i+1}) - 2u(\rho_i) . \quad (26)$$

and can be integrated numerically using, e.g. , a simple Euler scheme. Starting from an initial condition obtained by distributing at random N particles among the L sites of the lattice, we see in figure 7 that stochastic and mean-field dynamics agree very well, despite the fact that the mean-field approximation (by definition) neglects both noise and correlations. We infer from this that activated events and spatial correlations are not very important to understand the early-stage dynamics and we shall thus proceed using this more analytically-tractable mean-field framework.

\ddagger because the model is factorable there can be no correlations in the true steady-state

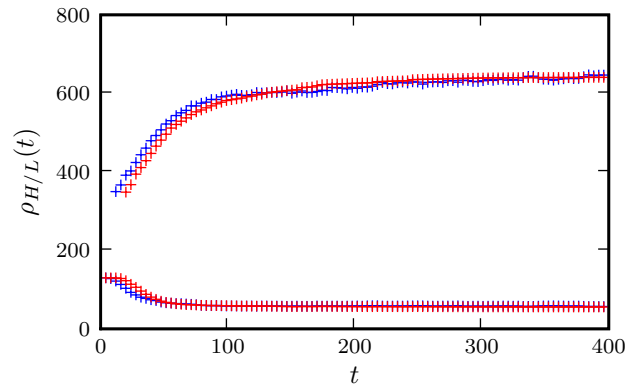


Figure 7. The evolution of the average high and low densities: using numerical simulations in the deterministic (red) and stochastic (blue) cases. Both simulations were run with random (Poissonian) initial conditions and with the parameters $\phi = 250$, $\lambda = 0.01$, $v_0 = 2.5$ and $\langle n \rangle = 130$. Although the agreement is not exact the qualitative behaviour is certainly similar. The slight lag between the stochastic and deterministic cases is due to activated events increasing the initial separation between high and low density sites and is not especially relevant to an understanding of the dynamics.

At early times, the number of condensates that are created depends strongly on the initial condition. For instance, starting the mean-field simulation from a flat profile superposed by a cosine wave leads to the creation of a condensate from each peak of the cosine wave, as shown in the left panel of figure 8. We shall first focus on the case where the number of condensates is thus controlled. As time goes on, the mass for the condensates is drawn from neighbouring sites, which suggests a model of this particular condensate-formation process as one in which sites have one of two time-dependent densities. Specifically each high-density site has a density $\rho_H(t)$, and is surrounded by a pair of low-density sites, both with density $\rho_L(t)$.

Within this simplification, the mean-field equations (26) reduce to

$$\dot{\rho}_H = 2[u(\rho_L) - u(\rho_H)]; \quad \dot{\rho}_L = p_{\ell c}[u(\rho_H) - u(\rho_L)] \quad (27)$$

where we used the fact that condensates have two low density neighbours whereas low density sites have a probability

$$p_{\ell c} = \frac{2n_c}{L - n_c} \quad (28)$$

of being next to a condensate. This simple approximation reproduces surprisingly well the mean-field dynamics for this particular initial condition, as can be seen from the right panel of figure 8. Inspection of equation (27) then gives a simple picture of what is happening: in the spinodal region $u'(n) < 0$, so that $u(\rho_L) - u(\rho_H) > 0$ and $\dot{\rho}_H$ is positive whereas $\dot{\rho}_L$ is negative. Consequently the high density will increase and the low density decrease. This continues as long as $u(\rho_H) < u(\rho_L)$ but stops at the first moment when $u(\rho_H) = u(\rho_L)$. This is indeed what happens during the simulation, as can be seen on figure 9.

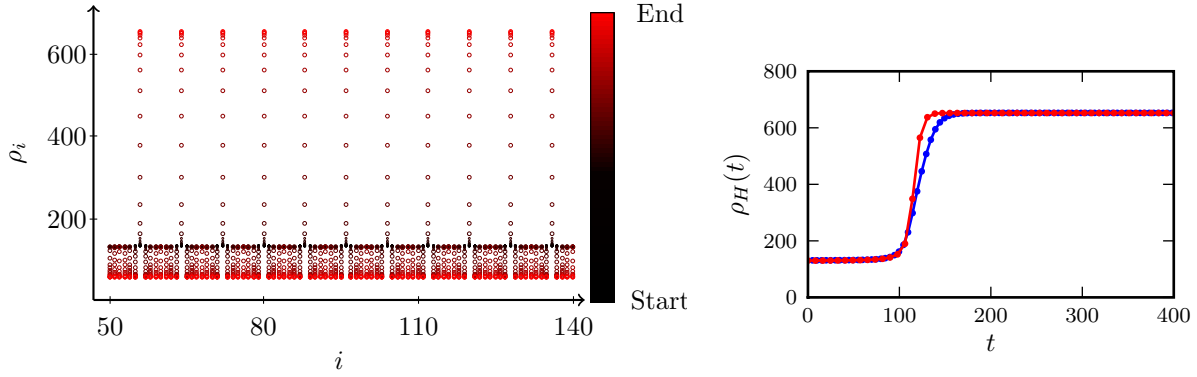


Figure 8. **Left:** Starting from an initial profile which is flat plus a cosine perturbation condensates grow at every peak (n_c is 35% higher than n_c^{eq}). Increasing time is represented by a change in colour. Parameters are $\lambda = 0.01$, $\phi = 250$, $v_0 = 2.5$ and $\langle n \rangle = 130$. **Right:** The mean-field stochastic simulations (red), as left, and the solutions to equations (27) (blue). The approximation to consider just two densities gives a sharper change but the end points are in excellent agreement.

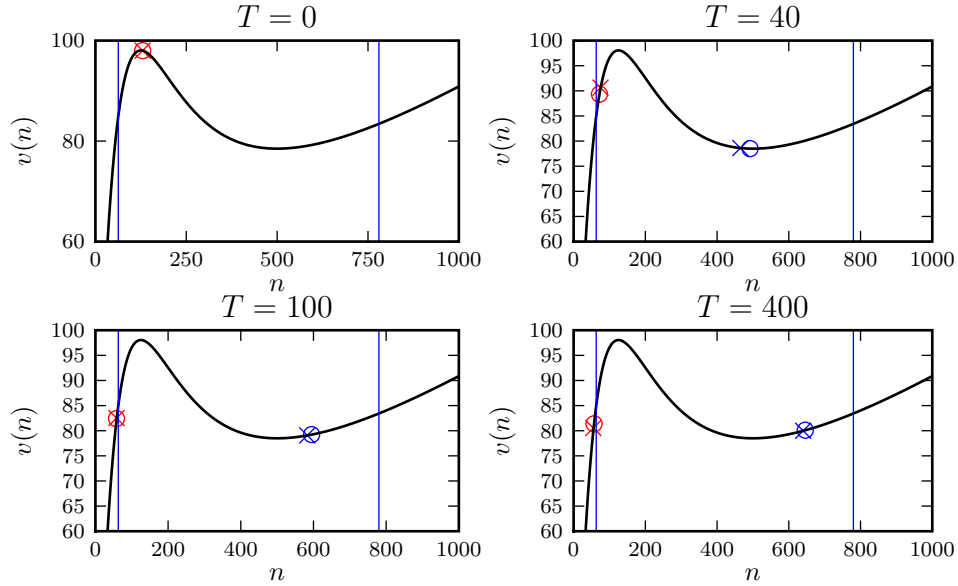


Figure 9. Snapshots of the hopping rate out of a site with n particles, $u(n) = v_0 n \exp(-\lambda \phi \arctan(n/\phi))$, for mean-field simulations (crosses) and stochastic ones (circles). The red and blue symbols represents the average high density and low density sites. The blue lines show the steady state values of ρ_H and ρ_L and the black line the function $u(n)$. At $t = 400$, $u(\rho_H) = u(\rho_L)$ and the condensate thus stop increasing. One must then wait for activated events in the stochastic simulations to get closer to the equilibrium values. The difference between stochastic and mean-field comes from the different number of condensates that results from the initial instability.

Note that according to the previous discussion, the average values of ρ_L and ρ_H at the end of the growth stage can be deduced from the initial number of condensates,

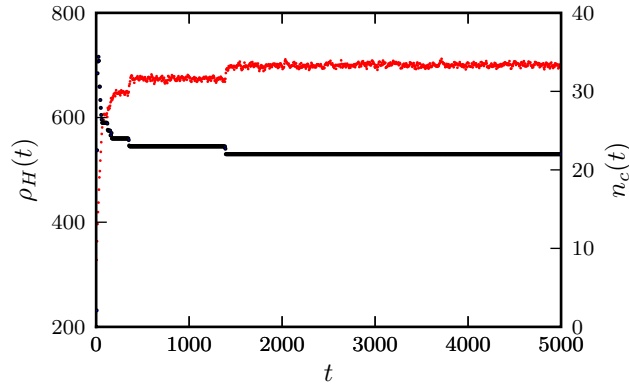


Figure 10. Typical evolution of the average density and number of condensate sites. Each steps correspond to the evaporation of a condensate that is redistributed over the other high density sites. The parameters of the simulations are $\lambda = 0.01$, $\phi = 250$ and $v_0 = 2.5$

using conservation of mass and requiring that $u(\rho_H) = u(\rho_L)$ (see equation (27))

$$n_c \rho_H + (L - n_c) \rho_L = N; \quad u(\rho_H) = u(\rho_L). \quad (29)$$

Starting simulations with different wavelength for the initial cosine perturbation indeed leads to density ρ_H and ρ_L predicted by (29) where n_c equals the number of peaks of the cosine wave.

For more general initial conditions or in the stochastic case, we have not been able to find a simple way to predict the number of condensates to be formed. The saturation of their growth once formed however follows the same rules as above, leading to high and low densities that in general differ from the steady-state ones (see figure 9). Further changes in the average densities are due to activated events which change the number of condensates and increase both ρ_L and ρ_H . These events are not captured by the mean-field approximation, and require a distinct analysis that is discussed in the next section.

3.2. Activated events and late-stage dynamics

To understand the late stage dynamics, which is mediated by stochastic nucleation and evaporation of condensates, we must investigate the activated crossing between the two wells in the grand-potential landscape shown in figure 2.

We have observed that during the late-stage relaxation, the total mass in the high-density sites remains approximately constant, despite the fact that the number of such sites changes over time. Thus, the average density per condensate increases in the step-wise fashion depicted in figure 10. When a condensate evaporates the excess density on that site is redistributed over the other high-density sites, thereby increasing their average density; conversely, when a new condensate forms, the average density decreases. Depending on whether the late stage starts with too many or too few condensates, evaporation or condensation will first dominate, before the two rates become closer and

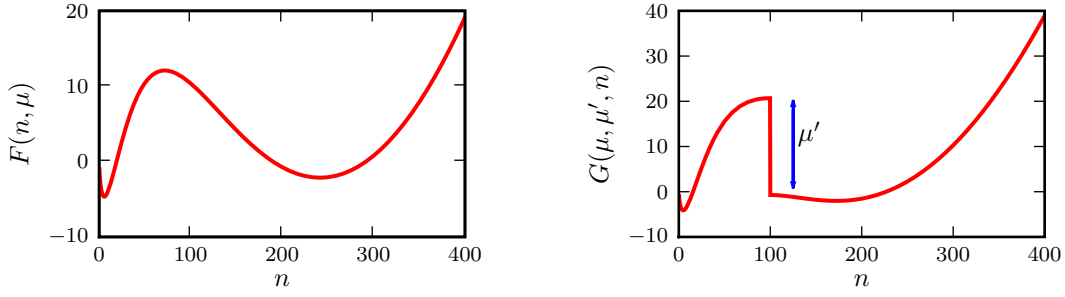


Figure 11. **Left:** the free energy per site for $v_0=2.5$, $\phi = 50$, $\lambda = 3/50$, $\langle n \rangle = 100$ at steady state, where there are 40% of condensate sites. **Right:** the free energy for the same microscopic rates and average occupancy but constrained to have 60% of condensate sites.

closer. Once evaporation and condensation of new condensates balance, the steady state described in section 2.3 is reached.

In the steady state, the rates of evaporation and condensation can be treated as a first passage problem in a grand potential landscape—as shown previously in section 2.3. Away from steady state, however, the rates are different from the equilibrium ones and depend in general on the dynamics. Numerically, we can measure the rates as a function of the fraction of condensates by recording how long the system spends in a given configuration with n_c condensates and averaging over many runs. We now show that these *nonequilibrium* rates can also be calculated by appealing to a fluctuation-dissipation-type argument and adapting the equilibrium formalism correspondingly. The key to this approach is to assume that when the number of condensates n_c is sufficiently close to its equilibrium value n_c^{eq} , the difference $n_c - n_c^{\text{eq}}$ could be due either to a spontaneous fluctuation (which we are observing) or *equivalently* to the application of a small field. To this end, we introduce a new ‘doubly-grand canonical’ ensemble that involves an additional chemical potential μ' conjugate to the number of condensates n_c . The corresponding partition function is then given by

$$\Xi = \sum_{N=0}^{\infty} \sum_{n_c=0}^L e^{\mu N} e^{\mu' n_c} Z_{L,N}. \quad (30)$$

Defining a condensate, as before, as any site containing more than n_{peak} particles, the partition function can be re-written as

$$\Xi = \sum_{n_c=0}^L e^{\mu' n_c} \left[\sum_{n=0}^{\infty} e^{-f(n) + \mu n} \right]^L = \sum_{n_c=0}^L \left[\sum_{n=0}^{\infty} \exp[-G(\mu, \mu', n)] \right]^L. \quad (31)$$

where we have introduced a new thermodynamic potential

$$G(\mu, \mu', n) = f(n) - \mu n - \mu' \theta(n - n_{\text{peak}}) \quad (32)$$

$G(\mu, \mu', n)$ can simply be obtained by introducing a step $-\mu'$ in the grand potential $F(\mu, n)$, see fig. 11. The new marginal probability to observe an occupancy n is then

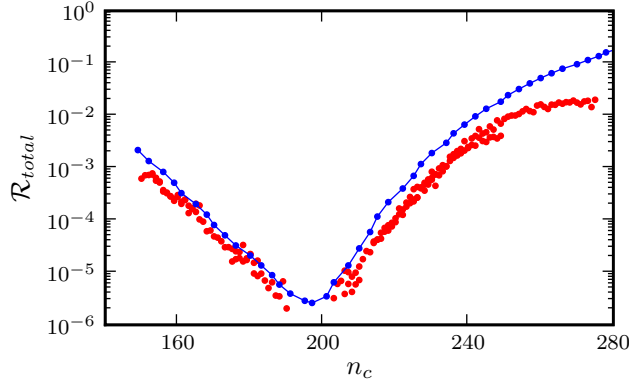


Figure 12. Red points show data from stochastic simulations of the microscopic dynamics. The time spent in a configuration with n_c condensates was averaged over multiple runs and the total rate to leave a given configuration was taken as the inverse of this average time. The blue curve shows the rate calculated by using equations (24) to determine the first-passage time to the peak and assuming a site at the peak is equally likely to fall into either well. All data calculated with $v_0 = 2.5$, $\phi = 50$, $\lambda = 3/50$, $\langle n \rangle = 100$ and on a lattice of length $L = 500$ sites.

given by

$$p(n|\mu, \mu') = \frac{e^{-G(\mu, \mu', n)}}{\Xi_1(\mu, \mu')}; \quad \text{where} \quad \Xi_1(\mu, \mu') = \sum_{n=0}^{\infty} e^{-G(\mu, \mu', n)} \quad (33)$$

To evaluate the rates of evaporation and condensation of the system in the presence of n_c condensates, one can thus compute numerically the values of μ and μ' such that

$$\sum_n n p(n|\mu, \mu') = N \quad \text{and} \quad \sum_n \theta(n - n_{\text{peak}}) p(n|\mu, \mu') = n_c \quad (34)$$

We can then compute the rate of evaporation or condensation from a configuration with N particles and n_c condensates exactly as in section 2.3 where $F(\mu, n)$ is now replaced by $G(\mu, \mu', n)$. To compare the results of these calculations with numerics we started 100 runs from $n_c = 275$ and 100 from $n_c = 155$. While these runs relaxed to the equilibrium value $n_c = 199$ we recorded the average time spent, $\tau(n_c)$, by the system for each intermediate value of n_c and approximated $\mathcal{R}_{\text{total}}(n_c) = 1/\tau(n_c)$. The results of these simulations are compared with the theoretical predictions in figure 12. The fact that the agreement between theory and numerics is not as good as in section 2.3 may be due to the fact that we have poorer statistics (100 runs against 10000). Nevertheless, we find that this ‘doubly- grand canonical’ construction provides remarkably good estimates for the evaporation and condensation rates: even where they are two orders of magnitude larger than in equilibrium, theory and simulations are still within a factor 2 of each other.

4. Conclusion

In this work we have identified a *saturated condensation* scenario that may occur within the zero-range process (ZRP). The steady state is characterized by an extensive number of finite-sized condensates, as opposed to a single macroscopic condensate that has previously been the focus of attention. Such a state may be brought about, for example, by a constraint on the total mass that may occupy a single site. We have determined the conditions on the hop rates within the ZRP that must be satisfied for saturated condensation to arise, and have investigated various aspects of the model's dynamics, both within and en-route to the steady state.

Our results complement existing treatments of the condensation dynamics for the version of the ZRP with a true condensation transition [9, 10, 11, 12, 13]. In those works the focus has mostly been on the late-time scaling behaviour of cluster size, and the characteristic timescales for evaporation and reformation of condensates. Within the model of saturated condensation discussed here, we have obtained a more complete account of the dynamics—including the nontrivial behaviour of the number of condensate sites—from very early times right through to the steady state. In particular, we find that the relaxation takes place in two stages: first, some number of condensate sites is dynamically selected which depends on the initial condition. These condense rapidly, but leave the system in an out-of-equilibrium state that slowly relaxes through activated evaporation and condensation events. A mean-field approximation proved reliable in analyzing the first stage, and a first-passage calculation conducted within a specially-constructed grand canonical ensemble well described the second.

There are, however, some discrepancies between these approximate analytical approaches and the full stochastic dynamics. For example, the rates we calculated for evaporation and condensation, both at equilibrium and away from the steady state, seem to overestimate those measured from the simulations. One plausible explanation for this inconsistency is the assumption that the neighbouring sites will have a constant density throughout the evaporation or condensation process. In fact the neighbouring sites are likely to have higher densities than otherwise during an evaporation and lower densities during a condensation. As both high and low density sites sit in regions where $v'(n) > 0$, an increase in density will increase the rate at which particles enter the evaporating site whilst a decrease in neighbouring density will decrease the rate at which particles enter a condensing site. Both these processes will have the effect of increasing the average duration of a condensation or evaporation event, and reducing the rates at which they happen. Hence the calculated evaporation and condensation rates would be larger than the real rates as measured.

More serious, perhaps, is the absence of a satisfactory explanation for the number of condensate sites formed from a uniform initial condition. One would anticipate that the early-time noise would play a major role in determining this number, although we have been unable to relate these two quantities directly. This we leave as a possible avenue for future work.

Finally, since we believe our model to reproduce more faithfully experimental situations such as the shaken granular gases than the traditional version of the ZRP, it would be very interesting to investigate both early- and late-time dynamics of the corresponding experiments [2].

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Appendix A. Solution of the first passage time problem

The first passage time from n particles to n_{peak} is denoted $T_{n,n_{\text{peak}}}$ and, in continuous time, is given by the solution to the equation

$$T_{n,n_{\text{peak}}} = dt + [1 - (u_L + u(n)) dt] T_{n,n_{\text{peak}}} + u(n) dt T_{n-1,n_{\text{peak}}} + u_L dt T_{n+1,n_{\text{peak}}} \quad (\text{A-1})$$

This equation states that the time to go from n to n_{peak} particles (l.h.s) is dt plus the time to go from the new number of particles, obtained after a time interval dt , to n_{peak} . With probabilities $u(n) dt$ and $u_L dt$, there are now $n - 1$ or $n + 1$ particles, while with probability $1 - (u_L + u(n)) dt$ there are still n particles, hence the three terms of the r.h.s. Equation (A-1) then reduces to

$$(u_L + u(n)) T_{n,n_{\text{peak}}} - u_L T_{n+1,n_{\text{peak}}} - u(n) T_{n-1,n_{\text{peak}}} = 1 \quad (\text{A-2})$$

with boundary condition $T_{n_{\text{peak}},n_{\text{peak}}} = 0$. Now, define the difference $d_n = T_{n,n_{\text{peak}}} - T_{n-1,n_{\text{peak}}}$ so that

$$u(n) d_n - u_L d_{n+1} = 1. \quad (\text{A-3})$$

Note that the evolution of the probability to find n particles at a site, $p(n|\mu)$, is given by

$$\frac{dp(n|\mu)}{dt} = u(n+1) p(n+1|\mu) + u_L p(n-1|\mu) - (u(n) + u_L) p(n|\mu) \quad (\text{A-4})$$

$$\equiv J_{n+1,n} - J_{n,n-1} \quad (\text{A-5})$$

where $J_{n+1,n} = u(n+1) p(n+1|\mu) - u_L p(n|\mu)$. At steady state the left hand side of this equation must equal zero, so the current, J must be constant. However, as $p(n < 0|\mu) = 0$ and $v_0 = 0$, $J_{-1,0} = 0$ and hence this constant must be zero. This implies that $u(n+1) p(n+1|\mu) = u_L p(n|\mu)$ so the solution to the homogeneous version of equation (A-3) is:

$$d_n = \frac{1}{u(n) p(n|\mu)}. \quad (\text{A-6})$$

To solve the inhomogeneous equation, then, we look for solutions of the form

$$d_n = \frac{c_n}{u(n) p(n|\mu)}. \quad (\text{A-7})$$

where c_n is to be determined. Substituting this expression back into equation (A-3) gives

$$-u_L \frac{c_{n+1}}{u(n+1) p(n+1|\mu)} + u(n) \frac{c_n}{u(n) p(n|\mu)} = 1. \quad (\text{A-8})$$

The detailed balance condition, $u(n+1) p(n+1|\mu) = u_L p(n|\mu)$, then implies

$$c_n - c_{n+1} = p(n|\mu), \quad (\text{A-9})$$

so that $c_n = \sum_{l=n}^{\infty} p(l|\mu)$ and the first passage time is given by

$$T_{n,n_{\text{peak}}} = \sum_{l=n_{\text{peak}}+1}^n \frac{1}{u(l) p(l|\mu)} \sum_{m=l}^{\infty} p(m|\mu). \quad (\text{A-10})$$